PROFILES OF MEAN TEMPERATURE AND CONCENTRATION IN TURBULENT FLOW OF A LIQUID IN A PIPE AND IN A BOUNDARY LAYER AT A PLATE

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A single equation is obtained which describes the profiles of mean temperature and concentration over the entire thickness of the internal region of a turbulent boundary layer at a wall. Theoretical and experimental data are compared for liquids with $0.026 \le \mathrm{Pr} \le 1000$.

Many papers have been devoted to a study of the mean temperature and mean concentration profiles [1-4, 7-20], but the results obtained do not provide an opportunity to describe the temperature and concentration profiles over the entire thickness of the internal region of a turbulent wall boundary layer in terms of a single relationship.

In the determination of temperature profiles, it is usually considered advisable to start from a method for the determination of the mean velocity profile, since the latter is satisfactorily investigated at the present time. The temperature and velocity profiles are, respectively, determined from

$$q_l = -(\lambda_l + \rho_l c_{pl} \varepsilon_q) \frac{dT}{dy}, \tag{1}$$

$$\tau_l = (\mu_l + \rho_l \varepsilon_\tau) \frac{du}{dy}. \tag{1a}$$

As a rule, in the determination of temperature and velocity profiles, the investigator divides the boundary layer into two or three structurally different regions, each of which has its own relationship for the determination of the turbulent transport coefficients ε_q and ε_τ.

Several of the relationships for ε_τ obtained by various authors for individual layers of a turbulent boundary layer are presented below.

Deissler [9], assuming a two-layer scheme for the boundary layer, gives the following expression for the region near the wall:

$$\varepsilon_\tau = n^2 uy \left[1 - \exp\left(-n^2 \frac{uy}{\nu}\right)\right]. \tag{2a}$$

For an intermediate layer, there was proposed [3]

$$\varepsilon_\tau \sim b\nu \left(\frac{y}{\delta}\right)^2. \tag{2b}$$

Prandtl obtained the following equation for a turbulent layer:

$$\varepsilon_\tau = \varkappa^2 y^2 \frac{du}{dy}. \tag{2c}$$

Here, n and b are constants; $\kappa = 0.4$ is the universal Karman constant.

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If the purpose is to obtain a single equation which describes the velocity and temperature profiles for the entire thickness of the internal region of a turbulent wall boundary layer, it is necessary to combine the velocity expressions given above and to change the form of the resultant relation so that it characterizes the temperature variation also. Interpolation relations for the turbulent coefficients of viscosity and thermal diffusivity which combine Eqs. (2a)-(2c) can be written, respectively, in the form

$$\varepsilon_\tau = \varkappa y v^* \left[1 - B\left(\frac{y}{\delta_h}\right)^2 f_1(\varkappa, \mu, y)\right], \tag{2d}$$

$$\varepsilon_q = \varkappa y v^* \left[1 - A\left(\frac{y}{\delta_V}\right)^2 f(\varkappa, \mu, \lambda, y)\right], \tag{2}$$

where A and B are constant coefficients; δ_h and δ_V are, respectively, the thicknesses of the hydrodynamic and thermal wall layers, where it is necessary to take into account the contribution from molecular and turbulent transport coefficients [4]. If one assumes a two-layer scheme for the boundary layer, $\delta_V = Pr^{-1/z}\delta_h$, where $z \geq 3$ [2]; $\delta_h = 11.5\ \nu/v^* = 11.5\delta^*$.

The functions $f(\kappa, \mu, \lambda, y)$ and $f_1(\kappa, \mu, y)$ must go to zero as the argument (y) increases in order that the molecular transport coefficients vanish from Eqs. (2) and (2d). Following [4, 9], therefore, we assume

$$f_1(\varkappa, \mu, y) = \exp\left(-\varkappa^2 \frac{v^* y}{\nu_W}\right), \tag{3a}$$

$$f(\varkappa, \mu, \lambda, y) = \mathrm{Pr}_l^a \exp\left(-\varkappa^2 \mathrm{Pr}_l^k \frac{v^* y}{\nu_W}\right). \tag{3}$$

For turbulent flow at a wall, $l = \kappa y$ and hence $y = l/\kappa$. Here l is the Prandtl mixing length which for velocity and temperature, respectively, is written in the form

$$l_u^2 = \frac{\mu_\tau}{\rho \dfrac{du}{dy}}, \tag{4a}$$

$$l_\tau^2 = \frac{\lambda_\tau T^*}{\rho c_p v^* \dfrac{dT}{dy}}. \tag{4}$$

Equations (4) and (4a) only take into account the contribution from turbulence. It is possible to ignore the effect of molecular transport coefficients for the greater portion of the layer, since the thickness of the turbulent layer is very much greater than the thickness of the molecular transport layer. Their values become comparable only upon approach to the wall. Therefore, in the wall region of the boundary layer ($y < \delta_V$ for temperature and $y < \delta_h$ for velocity), it is necessary to take into account the turbulent and molecular transport coefficients. Because of this, we write in place of μ_τ and λ_τ in Eqs. (4a) and (4) their effective values

$$\mu_{ef} = \mu_l + \mu_\tau, \quad \lambda_{ef} = \lambda_l + \lambda_\tau.$$

Using what has been presented above, Eqs. (2d) and (2) for ε_τ and ε_q take the forms

$$\varepsilon_\tau = \varkappa y v^* \left[1 - \alpha \frac{\mu_l + \mu_\tau}{(\delta^*)^2 \rho_l} \frac{\exp\left(-\varkappa^2 \dfrac{y v_*}{\nu_W}\right)}{\dfrac{du}{dy}}\right], \tag{5a}$$

$$\varepsilon_q = \varkappa y v^* \left[1 - \beta \frac{(\lambda_l + \lambda_\tau)\,\mathrm{Pr}_l^m}{(\delta^*)^2 \rho_l c_{p_l} v^*} \frac{\exp\left(-\varkappa^2 \mathrm{Pr}_l^k \dfrac{y v^*}{\nu_W}\right)}{\dfrac{dT}{dy}}\right], \tag{5}$$

where $\alpha = B/(11.5\kappa)^2$ and $\beta = A/(11.5\kappa)^2$ are constants and $m = a + (2/z)$.

Substituting the respective values of ε_q and ε_τ from Eqs. (5) and (5a) into Eqs. (1) and (1a), we obtain after transformation and integration under the conditions $q_l = q_W$ and

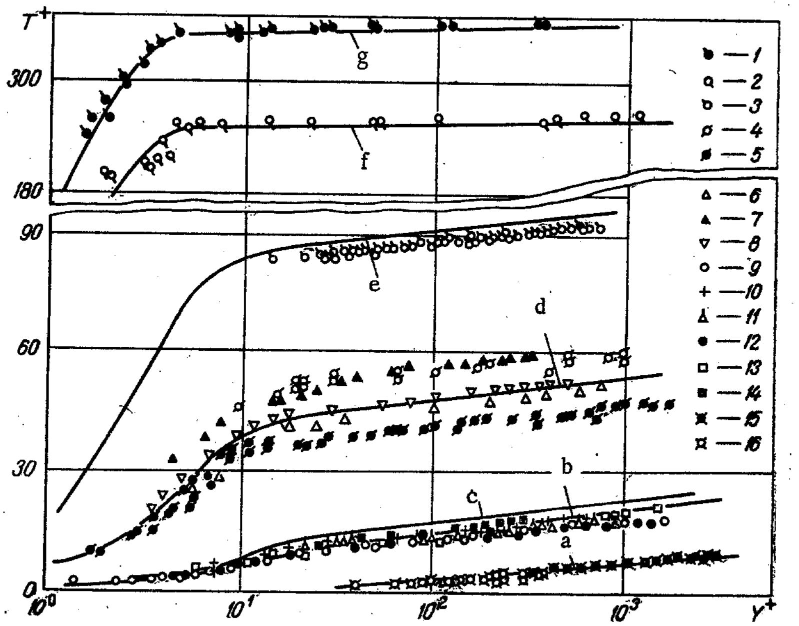

Fig. 1. Dimensionless temperature profile in flows of transformer oil, water, air, and liquid metal: 1) $Pr_l = 90$, $Pr_w = 72$ [7]; 2) $Pr_l = 67$, $Pr_w = 41$ [7]; 3) Pr = 14.3, Re = 16,400-25,200 [13]; 4) $Pr_l = 5.45$, $Pr_w = 4.9$ [7]; 5) Pr = 5.35 [8]; 6) Pr = 5.7, Re = 34,900 [13]; 7) Pr = 5.7, Re = 11,000 [13]; 8) Pr = 5.7, Re = 16,600 [13]; 9) Pr = 0.7 [8]; 10) Pr = 0.7 [12]; 11) Pr = 0.7 [13]; 12) Pr = 0.7 [10]; 13) Pr = 0.73, Re = 77,000 [16]; 14) Pr = 0.73, Re = 10,000 [16]; 15) Pr = 0.026, Re = 110,000-160,000 [13]; 16) Pr = 0.026 [14]. Solid curves are from Eq. (7) for k = 0.5 and n = 0.85: a) Pr = 0.026; b) Pr = 0.7; c) Pr = 1; d) Pr = 5.7; e) Pr = 14.3; f) Pr = 54; g) Pr = 81.

$\tau_l = \tau_w$ equations which describe the temperature and velocity profiles for the entire thickness of the internal region of a turbulent wall flow:

$$T^+ = \frac{1}{\varkappa} \ln\left(1 + \varkappa \frac{\nu_w}{\nu_l} \Pr_l Y^+\right) + \frac{\beta}{\varkappa^3} \Pr_l^n \left[1 - (1 + \varkappa^2 \Pr_l^k Y^+) \exp(-\varkappa^2 \Pr_l^k Y^+)\right], \tag{6}$$

$$U^+ = \frac{1}{\varkappa} \frac{\rho_w}{\rho_l} \ln\left(1 + \varkappa \frac{\nu_w}{\nu_l} Y^+\right) + \frac{\alpha}{\varkappa^3} \left[1 - (1 + \varkappa^2 Y^+) \exp(-\varkappa^2 Y^+)\right], \tag{6a}$$

where n = m − 2k.

The value α = 1/2 satisfies Eq. (6a) [5]. Considering that when Pr = 1 and flow is isothermal, the velocity and temperature profiles in a turbulent flow are identical, we assume β = α = 1/2. Then Eqs. (6) and (6a) take the forms

$$T^+ = 2.5 \ln\left(1 + 0.4 \frac{\nu_w}{\nu_l} \Pr_l Y^+\right) + 7.81 \Pr_l^n \left[1 - (1 + 0.16 \Pr_l^k Y^+) \exp(-0.16 \Pr_l^k Y^+)\right], \tag{7}$$

$$U^+ = 2.5 \frac{\rho_w}{\rho_l} \ln\left(1 + 0.4 \frac{\nu_w}{\nu_l} Y^+\right) + 7.81 \left[1 - (1 + 0.16 Y^+) \exp(-0.16 Y^+)\right]. \tag{7a}$$

Equation (7) will also describe the concentration profile if we write Sc in place of Pr. For isothermal flow, Eqs. (7) and (7a) for the velocity, temperature, and concentration profiles can be written in the general form

$$\vartheta^+ = 2.5 \ln(1 + 0.4 \Pr Y^+) + 7.81 \Pr^n \left[1 - (1 + 0.16 \Pr^k Y^+) \exp(-0.16 \Pr^k Y^+)\right], \tag{8}$$

where $\vartheta^+ = U^+$, and Pr = 1 for velocity, $\vartheta^+ = T^+$ for temperature, and Pr = Sc for concentration.

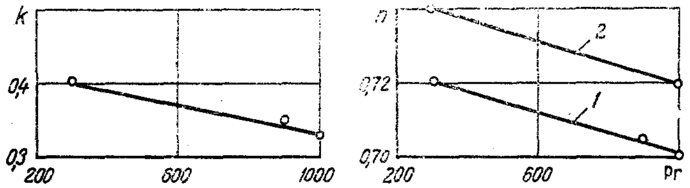

Fig. 2. Values of the exponents k and n in Eq. (11): 1) Re = 10,000; 2) Re = 30,000.

Experimental temperature data [7-20] satisfy the equations when $1/3 \leq k \leq 1/2$ and $3/2 < m < 4/2$. One can use the average values $k \approx 2/5$ and $m = 7/4$. However, it is necessary to take values of k and m somewhat greater than the average values for liquids with moderate Prandtl numbers and values close to, or somewhat below, the average values for liquids with very high Prandtl numbers. For liquids with $0.02 \leq Pr \leq 80$ [7-15], satisfactory agreement with experimental data is obtained for $k = 1/2$ and $n = 0.85$. A comparison between results calculated from Eq. (7) and experimental data for liquids with $0.02 \leq Pr \leq 80$ is presented in Fig. 1. The experimental data are given with respect to an average temperature $t_{av} = (1/2)(t_w + t_l)$. We therefore set $Pr_l = Pr_{av}$ and $\nu_w = \nu_l = 1$.

For Pr = 1 and isothermal flow, the velocity and temperature profiles are identical (curve c in Fig. 1). Experimental data for velocity are not shown in the figure.

Equations (7) and (7a), respectively, describe the temperature and velocity profiles in the boundary-layer region for constant thermal flux and tangential stress. They do not satisfy the natural condition $dT^+/dy^+ = 0$ and $dU^+/dY^+ = 0$ at the axis of a pipe. Therefore, Eqs. (7) and (7a) are not applicable for the central portion of a pipe and in the outer region of a boundary layer. These equations are only valid for the internal region of a turbulent wall flow. However, the value of the logarithm changes slowly with a change in the argument. Apparently, this fact provides an opportunity to use Eqs. (7) and (7a) for the central portion of a pipe also in approximate calculations.

The dimensionless temperature profile is often given in the form $(T - T_w)/(T_l - T_w) = f(Y^+)$. Then the temperature is determined from the equation

$$T^+ = \frac{T_w - T}{T_w - T_l}\,\frac{\sqrt{\xi/8}}{St}. \tag{9}$$

It is clear from Eq. (9) that the accuracy of a calculated value of the temperature depends on the error in the determination of St of Nu and ξ. If Nu and ξ are determined, respectively, by

$$Nu = \frac{\xi}{8} Re Pr^{0.4}, \quad \xi = 0.184 Re^{-0.2}, \tag{10}$$

we obtain

$$\frac{T - T_w}{T_l - T_w} = \frac{\sqrt{\xi/8}}{Pr^{0.6}} T^+. \tag{11}$$

Temperature values calculated from Eq. (11) for k = 0.5 and n = 0.85 differ somewhat from experimental data obtained for concentration and temperature at Pr = 300, 900, and 1000 [18, 19]. The calculated curve is in good agreement with the experimental data at Pr = 300, 900, and 1000 if the values of the exponents k and n are taken to be somewhat less than those for liquids with a moderate Prandtl number (Fig. 2). A comparison between the curve calculated from Eq. (11) and the experimental data is shown in Fig. 3 (values of k and n are determined from Fig. 2).

NOTATION

q, thermal flux density; τ, tangential stress; μ, ν, molecular coefficients of dynamic and kinematic viscosity; λ, λ_T, molecular and turbulent thermal conductivities; ρ, density; c_p, heat capacity; u, velocity; v^*, dynamic velocity; ε_T, μ_T, coefficients of turbulent kinematic and dynamic viscosity; ε_q, turbulent thermal diffusivity; δ^*, dynamic length; $T^* = q/\rho c_p v^*$, temperature scale; $T^+ = (T_w - T)/T^*$, dimensionless temperature; $U^+ = u/v^*$, dimensionless velocity; $Y^+ = Yv^*/\nu_w$, dimensionless coordinate; Re, Pr, Sc, Nu, Reynolds,

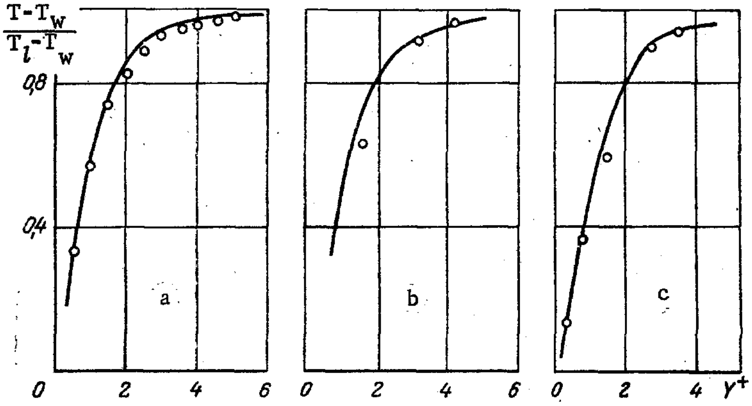

Fig. 3. Temperature and concentration profiles for large values of Pr and Sc: a) Sc = 900, Re = 9700 [19]; b) Pr = 300, Re = 10,000 [18]; c) Pr = 1000, Re = 30,000 [18]. Solid curves are from Eq. (11).

Prandtl, Schmidt, and Nusselt numbers; ξ, hydraulic resistance coefficient. Indices: w, l, quantities calculated with respect to the temperature of wall and liquid, respectively; ef, av, effective and average values of quantities, respectively.

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